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# THEORY OF NORMAL CONTACT OF RIGID BODIES

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*The theory of normal contact of rigid bodies with allowance for Coulomb friction is developed. The Boussinesq principle is generalized to contact problems with friction.*

**1. Governing Equations.** The equations of equilibrium, the strain-displacement relations, and Hooke's law are written in the form [1]

$$\sigma_{ij,j} + f_i = 0, \qquad e_{ij} = 0.5(u_{i,j} + u_{j,i}) = E^{-1}[(1+\nu)\sigma_{ij} - \nu\delta_{ij}\sigma_{kk}]. \tag{1.1}$$

Here $E$ is Young's modulus, $\nu$ is the Poisson ratio, $u_i$ are the displacements, $f_i$ are the volume forces, $\delta_{ij}$ are the Kronecker symbols, $e_{ij}$ are the strains, and $\sigma_{ij}$ are the stresses in the Cartesian coordinate system $x_i$; the subscript $i$ after a comma refers to partial differentiation with respect to $x_i$; summation is performed over repeated indices ($i$, $j$, $k = 1$, 2, 3).

**2. Work of Frictional Forces in Displacement Variations.** The general theory of normal contact of rigid bodies is developed on the basis of the results obtained in [2, 3] in solving the contact problems for a plate with an insert. We consider contact between two bodies $V$ and $\hat{V}$ bounded by the surfaces $S$ and $\hat{S}$, respectively. Let these surfaces consist of two parts $S = S_1 \cup S_2$ and $\hat{S} = \hat{S}_1 \cup \hat{S}_2$. The parts of $S_1$ and $\hat{S}_1$ that come in contact with one another upon loading and deformation of bodies are denoted by $S$ and $\hat{S}$; note that the zero forces are set at the points of zero contact $S_1$ and $\hat{S}_1$. At $S_2$ and $\hat{S}_2$, the boundary conditions can specify, for example, the displacements at one point and the forces at the others:

$$\boldsymbol{u} = \boldsymbol{u}^* \quad \text{on} \quad S_2' \text{ and } \hat{S}_2', \qquad \boldsymbol{p} = \boldsymbol{p}^* \quad \text{on} \quad S_2'' \text{ and } \hat{S}_2'', \tag{2.1}$$

where $\boldsymbol{u}^*$ and $\boldsymbol{p}^*$ are the specified displacement and force vectors, respectively; $S_2 = S_2' \cup S_2''$ and $\hat{S}_2 = \hat{S}_2' \cup \hat{S}_2''$.

According to the principle of possible displacements, in any variations of the displacements $\delta u_i$ and in the variations of appropriate strains $\delta e_{ij}$ the work of stresses in each body is equal to the work of the external forces applied to it:

$$\delta\Phi = \delta\Phi_V + \delta\Phi_1 + \delta\Phi_2, \qquad \delta\hat{\Phi} = \delta\hat{\Phi}_V + \delta\hat{\Phi}_1 + \delta\hat{\Phi}_2, \tag{2.2}$$

Here

$$\delta\Phi = \int\limits_V \sigma_{ij}\delta e_{ij}\,dV, \quad \delta\Phi_V = \int\limits_V \boldsymbol{f}\cdot\delta\boldsymbol{u}\,dV, \quad \delta\Phi_1 = \int\limits_{S_1} \boldsymbol{p}\cdot\delta\boldsymbol{u}\,dS_1, \quad \delta\Phi_2 = \int\limits_{S_2} \boldsymbol{p}\cdot\delta\boldsymbol{u}\,dS_2,$$

and $\boldsymbol{f}$ and $\delta\boldsymbol{u}$ are the vectors of volume forces and displacement variations. The quantities $\delta\hat{\Phi}$, $\delta\hat{\Phi}_V$, $\delta\hat{\Phi}_1$, and $\delta\hat{\Phi}_2$ for a body $\hat{V}$ are determined similarly.

For both contacting bodies, the work of stresses should be equal to the sum of the works of external forces and the work of frictional forces in the contact region in the variations of displacements $\delta\Phi_q$, i.e.,

$$\delta\Phi + \delta\hat{\Phi} = \delta\Phi_V + \delta\hat{\Phi}_V + \delta\Phi_2 + \delta\hat{\Phi}_2 + \delta\Phi_q. \tag{2.3}$$

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Substituting $\delta\Phi$ and $\delta\hat{\Phi}$ from (2.2) into (2.3), we obtain the equality

$$\delta\Phi_q = \int\limits_{S_1} \boldsymbol{p}\cdot\delta\boldsymbol{u}\,dS_1 + \int\limits_{\hat{S}_1} \hat{\boldsymbol{p}}\cdot\delta\hat{\boldsymbol{u}}\,d\hat{S}_1. \tag{2.4}$$

The vectors $\hat{\boldsymbol{p}}$ and $\delta\hat{\boldsymbol{u}}$ are related to $\hat{S}_1$. Equality (2.4) holds irrespective of the form of boundary conditions at $S_2$ and $\hat{S}_2$ and is used below to formulate the boundary conditions at $S_1$ and $\hat{S}_1$.

**3. Contact Boundary Friction-Independent Conditions.** We now set the surface $\Omega$ which is close to the surfaces $S_1$ and $\hat{S}_1$ and define the coordinate system $\xi^\alpha$ on it. At each point of $\Omega$, we restore the normal, find the points at which this surface intersect the surfaces $S_1$ and $\hat{S}_1$, and assign them the same coordinates $\xi^\alpha$ as those at the point considered on $\Omega$. We assume that a one-to-one correspondence between the sets of all points on $\Omega$, $S_1$, and $\hat{S}_1$ is established by these triples of points, which lie on the same normal to $\Omega$, and a coordinate system $\xi^\alpha$ that is common for $\Omega$, $S_1$, and $\hat{S}_1$ is found.

At the points with the same coordinates $\xi^\alpha$ and the radius-vectors $\boldsymbol{r}$, $\boldsymbol{R} = \boldsymbol{r} + h\boldsymbol{n}$, and $\hat{\boldsymbol{R}} = \boldsymbol{r} + \hat{h}\boldsymbol{n}$, the metric tensors $a_{\alpha\beta}$, $A_{\alpha\beta}$, and $\hat{A}_{\alpha\beta}$ on $\Omega$, $S_1$ and $\hat{S}_1$ are, respectively, connected by the following relations [4]:

$$A_{\alpha\beta} = a_{\alpha\beta} - 2hb_{\alpha\beta} + h_{,\alpha}h_{,\beta} + h^2 b_{\alpha\omega}b^\omega_\beta, \quad \hat{A}_{\alpha\beta} = a_{\alpha\beta} - 2\hat{h}b_{\alpha\beta} + \hat{h}_{,\alpha}\hat{h}_{,\beta} + \hat{h}^2 b_{\alpha\omega}b^\omega_\beta. \tag{3.1}$$

Here $\boldsymbol{n}$ and $\boldsymbol{r}_\alpha$ are the unit normal and the basis vectors tangent to $\Omega$, respectively; we note that the vector $\boldsymbol{n}$ is considered directed from $S_1$ to $\hat{S}_1$ everywhere on $\Omega$, $h$ and $\hat{h}$ are reckoned along the normal to $\Omega$ and are in absolute magnitude equal to the distances from $S_1$ and $\hat{S}_1$ to $\Omega$, and $b_{\alpha\beta}$ are the coefficients of the second quadratic form of $\Omega$ ($b_{\alpha\beta} = -\boldsymbol{n}_{,\alpha}\cdot\boldsymbol{r}_\beta$); the subscript $\alpha$ after a comma denotes differentiation with respect to $\xi^\alpha$ ($\alpha$, $\beta$, $\omega = 1, 2$). It follows from (3.1) that for the matrices $A_{\alpha\beta}$ and $\hat{A}_{\alpha\beta}$ to be nondegenerate and positive, it suffices to require that the ratios of $h$ and $\hat{h}$ to the curvature radii of $\Omega$ and the pair-by-pair products of their derivatives be small compared to the quantities $a_{\alpha\beta}$. In particular, one can use the surface $S_1$ or $\hat{S}_1$ as $\Omega$.

The equality of the radius-vectors of any points on $S_1$ and $\hat{S}_1$ with coordinates $\xi^\alpha$ and $\hat{\xi}^\alpha$, respectively, which come in contact owing to the displacements $\boldsymbol{u}$ and $\hat{\boldsymbol{u}}$, is represented in the form

$$(\boldsymbol{r} + h\boldsymbol{n} + \boldsymbol{u})_{\xi^\alpha} = (\boldsymbol{r} + \hat{h}\boldsymbol{n} + \hat{\boldsymbol{u}})_{\hat{\xi}^\alpha}. \tag{3.2}$$

Here the subscripts $\xi^\alpha$ and $\hat{\xi}^\alpha$ refer to the coordinates of the points at which the quantities in brackets are calculated. Taking into account the small displacements and, hence, the small differences between the coordinates $\Delta\xi^\alpha = \hat{\xi}^\alpha - \xi^\alpha$, we expand the right part of (3.2) in a Taylor series at the point $\xi^\alpha$, discarding the products of $\Delta\xi^\alpha$ by the derivatives of displacements and all terms containing powers of $\Delta\xi^\alpha$ higher than the first power. We obtain the relation $\boldsymbol{u} = \hat{\boldsymbol{u}} + v_3\boldsymbol{n} + v_\alpha\boldsymbol{r}^\alpha$, in which the vectors of displacements $\boldsymbol{u}$ on $S_1$ and $\hat{\boldsymbol{u}}$ on $\hat{S}_1$ are taken at the points having the same coordinates $\xi^\alpha$. The differences between the components of these displacements $v_3 = (\boldsymbol{u} - \hat{\boldsymbol{u}})\cdot\boldsymbol{n} = c + \hat{h}_{,\beta}\Delta\xi^\beta$ and $v_\alpha = (\boldsymbol{u} - \hat{\boldsymbol{u}})\cdot\boldsymbol{r}_\alpha = (a_{\alpha\beta} - \hat{h}b_{\alpha\beta})\Delta\xi^\beta$ ($\alpha = 1, 2$) should be small, of the order of the displacements themselves. The quantity $c = \hat{h} - h \geqslant 0$ is a distance (gap) between $S_1$ and $\hat{S}_1$ measured along the normal to $\Omega$. If one takes $\Omega$ as $\hat{S}_1$, one has $\hat{h} = 0$ and $v_3 = c$ and $v^\alpha = \Delta\xi^\alpha$ ($\alpha = 1$ and $2$).

Ignoring the term $\Delta\hat{h} = \hat{h}_{,\beta}\Delta\xi^\beta$ in the expression for $v_3$, which is approximately equal to the difference between the values of $\hat{h}$ at the points $\hat{\xi}^\beta$ and $\xi^\beta$ and small compared with the magnitude of the gap $c$, we set $v_3 = c$. Now $v_3$ is a known function of the coordinates of only one point $\xi^\alpha$. On the contact-free surfaces $S_1$ and $\hat{S}_1$, as the nonpenetration condition we require the fulfillment of the inequality $v_3 \leqslant c$ for each pair of their points lying on one normal to $\Omega$ and having the same coordinates $\xi^\alpha$.

In the presence of friction, the differences between the tangential displacements $v_\alpha$ can depend on the history of loading and interaction between the bodies. Slippage of $S_1$ and $\hat{S}_1$ relative to each other and the change of the pairs of contacting points are taken into account by means of $v_\alpha$ in the contact region. The partial derivatives of $v_\alpha$ with respect to the loading (time) parameter $\tau$ are the slip velocities

$\dot{v}_\alpha = (a_{\alpha\beta} - \hat{h}b_{\alpha\beta})\vartheta^\beta$ (the dot denotes differentiation with respect to $\tau$; $\vartheta^\beta = d\hat{\xi}^\beta/d\tau$). If $\dot{v}_\alpha = 0$, we have $\vartheta^\alpha = 0$; the pair of contacting points does not change, and attachment occurs.

With $v_\alpha = (\boldsymbol{u} - \hat{\boldsymbol{u}}) \cdot \boldsymbol{r}_\alpha$ calculated, for each point $\xi^\alpha$ on $S_1$, one can approximate the coordinates of the contacting point $\hat{\xi}^\alpha$ on $\hat{S}_1$ from the equations $v_\alpha = (a_{\alpha\beta} - \hat{h}b_{\alpha\beta})\Delta\xi^\beta$ ($\alpha = 1$ and 2). If one ignores the terms $\hat{h}b_{\alpha\beta}$ in these equations as small quantities of the order of the ratios of $\hat{h}$ to the curvature radii of $\Omega$, we obtain $\hat{\xi}^\alpha = \xi^\alpha + v^\alpha$ ($\alpha = 1$ and 2).

We pass to integration in (2.4) over the coordinates $\xi^\alpha$ common for $\Omega$, $S_1$, and $\hat{S}_1$ and use the relation between the magnitudes of the elementary sites $dS_1 = \gamma d\hat{S}_1$ ($\gamma = A^{1/2}\hat{A}^{-1/2} > 0$; here $A$ and $\hat{A}$ are the determinants of the metric tensors $A_{\alpha\beta}$ and $\hat{A}_{\alpha\beta}$, respectively). Let us find the contact regions $\Omega_1$ on $\Omega$ and the free unloaded edge $\Omega_2$ ($\Omega = \Omega_1 \cup \Omega_2$). Let attachment of $\delta\boldsymbol{u} = \delta\hat{\boldsymbol{u}}$ occur everywhere on $\Omega_1$ in the displacement variation in the contact region. Then, the work of frictional forces is equal to zero:

$$\delta\Phi_q = 0 = \int_{\Omega_1} (\gamma\boldsymbol{p} + \hat{\boldsymbol{p}}) \cdot \delta\hat{\boldsymbol{u}}\, d\hat{S}_1 + \int_{\Omega_2} (\gamma\boldsymbol{p} \cdot \delta\boldsymbol{u} + \hat{\boldsymbol{p}} \cdot \delta\hat{\boldsymbol{u}})\, d\hat{S}_1.$$

In these integrals, we equate the coefficients of arbitrary variations of displacements $\delta\hat{\boldsymbol{u}}$ on $\Omega_1$ and $\delta\boldsymbol{u}$ and $\delta\hat{\boldsymbol{u}}$ on $\Omega_2$ to zero. We find that $\hat{\boldsymbol{p}} = -\gamma\boldsymbol{p}$ on $\Omega_1$ and $\hat{\boldsymbol{p}} = \boldsymbol{p} = 0$ on $\Omega_2$. We introduce the expansions of the force vectors into the normal and the tangents to $\Omega$: $\boldsymbol{p} = p_3\boldsymbol{n} + p_\alpha\boldsymbol{r}^\alpha$ and $\hat{\boldsymbol{p}} = \hat{p}_3\boldsymbol{n} + \hat{p}_\alpha\boldsymbol{r}^\alpha$. In the contact region, owing to pressing of the surfaces $S_1$ and $\hat{S}_1$ against each other and the directedness of the vector $\boldsymbol{n}$ from $S_1$ to $\hat{S}_1$, the inequalities $p_3 < 0$ and $\hat{p}_3 = -\gamma p_3 > 0$ should hold. The tangential components of the forces $\boldsymbol{q} = p_\alpha\boldsymbol{r}^\alpha$ and $\hat{\boldsymbol{q}} = \hat{p}_\alpha\boldsymbol{r}^\alpha = -\gamma\boldsymbol{q}$ are the frictional forces on $S_1$ and $\hat{S}_1$ referred to a unit area of these surfaces. The coefficient $\gamma$ depends on the dimensions of the elementary sites coming in contact on $S_1$ and $\hat{S}_1$ and on the slope of them to each other in the initial undeformed state. On the free sites of $S_1$ and $\hat{S}_1$, in the region of $\Omega_2$ we have $\hat{\boldsymbol{q}} = \boldsymbol{q} = 0$.

Thus, we have the boundary conditions

$$v_3 = c, \quad \hat{p}_i = -\gamma p_i, \quad p_3 < 0 \text{ on } \Omega_1, \qquad \hat{p}_i = p_i = 0, \quad v_3 \leqslant c \text{ on } \Omega_2, \tag{3.3}$$

where $i = 1$, 2, and 3. These conditions are formulated irrespective of the properties of $S_1$ and $\hat{S}_1$; below, they are supplemented by the boundary conditions that take into account the action of friction. In [5–7], the boundary conditions in the contact region are also formulated for the pairs of points lying on the same normal to the specified surface; however, equality (3.2) is approximated differently.

We now replace the variations in (2.4) by displacement velocities. With allowance for (3.3) and $\dot{v}_3 = 0$ on $\Omega_1$, we obtain the expressions for the frictional-force work power at the slip velocities

$$\dot{\Phi}_q = \int_{\Omega_1} \boldsymbol{q} \cdot \dot{\boldsymbol{v}}\, dS_1 = -\int_{\Omega_1} \hat{\boldsymbol{q}} \cdot \dot{\boldsymbol{v}}\, d\hat{S}_1,$$

where $\dot{\boldsymbol{v}} = \dot{v}_\alpha\boldsymbol{r}^\alpha$. The densities $Q = \boldsymbol{q} \cdot \dot{\boldsymbol{v}}$ and $\hat{Q} = -\hat{\boldsymbol{q}} \cdot \dot{\boldsymbol{v}} = \gamma Q$ should be nonpositive: $Q \leqslant 0$, $\hat{Q} \leqslant 0$, and $\dot{\Phi}_q \leqslant 0$. On each contact surface, the velocities of its sliding relative to the other surface and the frictional forces are directed oppositely. With opposite sign, the quantities $-\dot{\Phi}_q$, $-Q$, and $-\hat{Q}$ are the power of energy scattering for friction and its densities per unit area of $S_1$ and $\hat{S}_1$.

**4. Contact Problems with Allowance for Coulomb Friction.** Let friction obeys the Coulomb law [5] in the contact region $\Omega_1$; then the ratio between the moduli of the vectors of tangent and normal forces should not exceed the value of the friction coefficient $\mu$; therefore, we should have $p_3 < 0$ and $F = |\boldsymbol{q}| + \mu p_3 \leqslant 0$. The equality $F = 0$ determines the minimum admissible angle of slope of the force vector $\boldsymbol{p}$ to the surface $\Omega$.

Slippage of the surfaces $S_1$ and $\hat{S}_1$ relative to each other in the contact region with velocity $\dot{\boldsymbol{v}}$ with friction can occur only if the angle of slope of the force vector $\boldsymbol{p}$ to the surface $\Omega$ reaches the minimum admissible value and the frictional-force work power density is nonpositive ($Q \leqslant 0$). On each surface $S_1$ and $\hat{S}_1$, the velocities of its slip relative to the other surface and the frictional forces have opposite directions ($\dot{\boldsymbol{v}} = \chi\boldsymbol{q}$ and $\chi \leqslant 0$). The modulus of the vector $|\dot{\boldsymbol{v}}|$ is not restricted by the friction law and can be arbitrary, irrespective of the values of $|p_3|$ and $|\boldsymbol{q}|$. At the points on $\Omega_1$ at which the slip conditions do not hold, there is attachment $\dot{\boldsymbol{v}} = 0$.

Based on the aforesaid, we arrive at the boundary conditions

$$\begin{gathered} v_3 = c, \quad \dot{v}_\alpha = 0, \quad \hat{p}_i = -\gamma p_i, \quad p_3 < 0, \quad F < 0 \text{ on } \Omega_1', \\ v_3 = c, \quad F = 0, \quad \dot{v}_\alpha = \chi p_\alpha, \quad \hat{p}_i = -\gamma p_i, \quad p_3 < 0, \quad \chi \leqslant 0 \text{ on } \Omega_1'', \\ p_i = \hat{p}_i = 0, \quad v_3 \leqslant c \text{ on } \Omega_2. \end{gathered} \tag{4.1}$$

Here the normal and tangential to $\Omega$ components of the vectors of displacements and forces $\boldsymbol{u}$ and $\boldsymbol{p}$ on $S_1$ and $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{p}}$ on $\hat{S}_1$ are taken at the points lying on the same normal to $\Omega$ and having the same coordinates $\xi^\alpha$ as at the point considered on $\Omega$; $v_3 = (\boldsymbol{u} - \hat{\boldsymbol{u}}) \cdot \boldsymbol{n}$, $v_\alpha = (\boldsymbol{u} - \hat{\boldsymbol{u}}) \cdot \boldsymbol{r}_\alpha$ ($i = 1$, 2, and 3; $\alpha = 1$ and 2). Attachment occurs on $\Omega_1'$ and also at the points on $\Omega_1''$, where $\chi = 0$. There is slip at the remaining points on $\Omega_1''$, and the frictional-force work power density at slip velocities is nonpositive. The inequality $v_3 \leqslant c$ holds everywhere on $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 = \Omega_1' \cup \Omega_1''$.

In contrast to [2], in (4.1), the possibility that the inequality $\dot{F} < 0$, which assumes discontinuous variation in $\dot{F}$ with time on $\Omega_1''$ is satisfied, is ignored. In [2, 3], the case $\dot{F} < 0$ on $\Omega_1''$ is not realized in numerical solutions of the problem. In other aspects, expressions (4.1) are a generalization of the boundary conditions given in [2, 3] for a plate with an insert.

The partitions $\Omega = \Omega_1 \cup \Omega_2$ and $\Omega_1 = \Omega_1' \cup \Omega_1''$ are determined only with the use of the quantities $p_3$ and $F$ at the current moment of time. These values and the regions $\Omega_1'$, $\Omega_1''$, and $\Omega_2$ and the domains of solution of the problems as a whole can depend on the history of loading of the bodies and attachment and slip of the contacting surfaces relative to each other. The problems subject to boundary conditions (4.1) should be solved with allowance for the loading history.

It is noteworthy that the boundary conditions (4.1) are satisfied at the pairs of points lying on one normal to $\Omega$ that are specified according to the proposed approximate formation of the problem, rather than at the contacting points. The possibility of slippage of the surfaces $S_1$ and $\hat{S}_1$ in the contact region relative to each other and the possibility of changing the pairs of contacting points is taken into account by means of the differences between the tangential displacements $v_\alpha$. Calculating $v_\alpha$, one can approximate the coordinates of the points of contact.

When there is no friction, supplementing (3.3) with the expressions in which the tangent forces in the contact region are zero, we obtain the following boundary conditions:

$$v_3 = c, \quad p_\alpha = \hat{p}_\alpha = 0, \quad \hat{p}_3 = -\gamma p_3, \quad p_3 < 0 \text{ on } \Omega_1, \quad p_i = \hat{p}_i = 0, \quad v_3 \leqslant c \text{ on } \Omega_2 \tag{4.2}$$

($i = 1$, 2, and 3; $\alpha = 1$ and 2); note that these conditions follow from (4.1) for $\mu = 0$. The regions $\Omega_1$ and $\Omega_2$ are determined from the solution of the problem. In the problem for Eqs. (1.1) subject to boundary conditions (2.1) and (4.2). we have a unique solution. The functional of the total potential energy

$$\Psi = \int\limits_{V \cup \hat{V}} \Big(\frac{1}{2}\sigma_{ij}e_{ij} - f_i u_i\Big)\, dV - \int\limits_{S_2'' \cup \hat{S}_2''} \boldsymbol{p}^* \cdot \boldsymbol{u}\, dS$$

reaches the minimum on it in the space of displacements which are subject to the boundary conditions $\boldsymbol{u} = \boldsymbol{u}^*$ on $S_2'$ and $\hat{S}_2'$ and the nonpenetration condition $v_3 \leqslant c$ everywhere on $\Omega = \Omega_1 \cup \Omega_2$.

**5. Generalization of the Boussinesq Principle.** In many studies (see, for example, [5, 8]), the Boussinesq principle is used to find the boundaries of the contact zones for frictionless problems; according to this principle, in the contact region $\Omega_1$, in approaching the boundary with the free-edge region $\Omega_2$ the normal force $p_3$ should tend to zero. In the presence of friction, the principle can be supplemented by the assumption that in the attachment region $\Omega_1'$, the force function $F$ tends to zero as the boundary with the slip region $\Omega_1''$ is approached. If $\Omega_1'$ is adjacent to $\Omega_2$, one can expect that the functions $p_3$ and $F$ will tend to zero in $\Omega_1'$ in approaching the boundary with $\Omega_2$. Satisfaction of the formulated conditions ensures continuity of the required functions upon passage through the contact boundary. This generalized principle is applied in [2, 3] to the solution of the contact problems for a plate with an insert by the finite-element method.

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